Classification of unit-vector fields in convex polyhedra with tangent boundary conditions

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Abstract

A unit-vector field $\mathbf{n}$ on a convex three-dimensional polyhedron $\bar{P}$ is tangent if, on the faces of $\bar{P}$, $\mathbf{n}$ is tangent to the faces. A homotopy classification of tangent unit-vector fields continuous away from the vertices of $\bar{P}$ is given. The classification is determined by certain invariants, namely edge orientations (values of $\mathbf{n}$ on the edges of $\bar{P}$), kink numbers (relative winding numbers of $\mathbf{n}$ between edges on the faces of $\bar{P}$), and wrapping numbers (relative degrees of $\mathbf{n}$ on surfaces separating the vertices of $\bar{P}$), which are subject to certain sum rules. Another invariant, the trapped area, is expressed in terms of these. One motivation for this study comes from liquid crystal physics; tangent unit-vector fields describe the orientation of liquid crystals in certain polyhedral cells.

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1 Introduction

A unit-vector field $\mathbf{n}$ on a convex polyhedron $\bar{P} \subset \mathbb{R}^3$ is a map from $\bar{P}$ to the unit sphere $S^2 \subset \mathbb{R}^3$. $\mathbf{n}$ is said to satisfy *tangent boundary conditions*, or, more simply, to be tangent, if, on the faces of $\bar{P}$, $\mathbf{n}$ is tangent to the faces. Tangent boundary conditions imply that, on the edges of $\bar{P}$, $\mathbf{n}$ is parallel to the edges, and therefore that $\mathbf{n}$ is necessarily discontinuous at the vertices. Let $P \subset \mathbb{R}^3$ denote $\bar{P}$ without its vertices (thus $\bar{P}$ is the closure of P). Let $C^0(P)$ denote the space of continuous tangent unit-vector fields on P. We have the usual notion of homotopic equivalence in $C^0(P)$; two maps $\mathbf{n}$, $\mathbf{n}' \in C^0(P)$ are homotopic, denoted $\mathbf{n} \sim \mathbf{n}'$, if there exists a continuous map $\mathbf{H} : P \times [0,1] \to S^2; (\mathbf{x}, t) \mapsto \mathbf{H}_t(\mathbf{x})$, such that $\mathbf{H}_t$ is tangent and $\mathbf{H}_0 = \mathbf{n}$, $\mathbf{H}_1 = \mathbf{n}'$.

Here we classify unit-vector fields in $C^0(P)$ up to homotopy. The paper is organised as follows. To a unit-vector field $\mathbf{n} \in C^0(P)$ we associate certain homotopy invariants, which we call *edge orientations*, *kink numbers*, and *wrapping numbers* (Section 3). Edge orientations are just the values of $\mathbf{n}$ on the edges of P (as noted above, there are two possible values, differing by a sign). Kink numbers are the integer-valued relative winding numbers of $\mathbf{n}$ between adjacent edges along a face of P. Wrapping numbers are the integer-valued relative degrees of $\mathbf{n}$ on planar surfaces which separate one vertex of P from the others. The continuity of $\mathbf{n}$ imposes sum rules on the kink numbers and wrapping numbers. In Section 4 we construct representative maps for each of the allowed sets of values of the invariants. In Section 5 we show that an arbitrary map $\mathbf{n} \in C^0(P)$ is homotopic to the reference map with the same values of the invariants. One part of the proof, concerning homotopies on the boundary of P, is deferred to Section 6.

We remark that it is the tangent boundary conditions which substantially determine the classification. In contrast, continuous unit-vector fields satisfying *fixed* boundary conditions – for simplicity, imagine $\mathbf{n}$ to be constant on the boundary of P – are equivalent to continuous maps of S^3 (the unit ball in $\mathbb{R}^3$ with boundary points identified) to S^2. As is well known, such maps are classified by the Hopf invariant. The absence of a Hopf invariant for tangent unit-vector fields is due to the vertex discontinuities.

The problem considered here is part of a study of extremals of the energy functional

$$E = \int_P \sum_{j,k=1}^{3} \partial_j n_k \partial_j n_k \, d^3r \tag{1}$$

defined on tangent unit-vector fields in $C^0(P)$ with square-integrable derivative. Lower bounds for the energy in terms of the invariants, along with

upper bounds for the case where P is a cube, will be reported elsewhere [6].

The study of these extremal maps is motivated in part by the study of liquid crystals in polyhedral cells. In the continuum limit, the average local molecular orientation of a uniaxial nematic liquid crystal may be described by a unit-vector field $\mathbf{n}$ (but see below). The energy of a configuration $\mathbf{n}$ – the so-called Frank energy – reduces, in a certain approximation (the so-called one-constant approximation), to the expression (1) [2]. Polyhedral liquid crystal cells can be manufactured so that $\mathbf{n}$ is approximately tangent to the cell surfaces. The homotopy type of $\mathbf{n}$ determines, at least in part, the optical properties of the liquid crystal, and is relevant to the design of liquid crystal displays [8].

In fact, the local orientation of a liquid crystal is only determined up to a sign, as antipodal orientations are physically equivalent. Therefore, it is properly described by a director field, a map from P to the real-projective plane RP^2, rather than a unit-vector field. However, because P is simply connected, a continuous director field on P can be lifted to a continuous unit-vector field. The lifted unit-vector field is determined up to an overall sign. As is shown in Section 3, $+\mathbf{n}$ and $-\mathbf{n}$ belong to distinct homotopy classes; their kink numbers are the same, but their edge orientations and wrapping numbers differ by a sign. By identifying these pairs of homotopy classes, we obtain a classification of continuous tangent director fields on P.

Twice-differentiable extremals of (1) are examples of harmonic maps. Harmonic maps between Riemannian polyhedra have been studied by Gromov & Schoen [5] and Eells & Fuglede [3]. In the case where the target manifold has nonpositive Riemannian curvature, results concerning the existence, uniqueness and regularity of solutions of the Euler-Lagrange equations have been established. Harmonic unit-vector fields in $\mathbb{R}^3$ have been studied by Brezis *et al* [1], also in connection with liquid crystals. The topological classification of liquid crystal configurations in $\mathbb{R}^3$ as well as in domains with smooth boundary has been extensively discussed – see, eg, Mermin [7], de Gennes and Prost [2], and Kléman [4].

We remark that the homotopy classification of tangent unit-vector fields on P may be regarded as the decomposition of $C^0(P)$ into its path-connected components with respect to the *compact-open topology*. The compact-open topology on $C^0(P)$ is generated by sets $[K, U]$, defined for compact $K \subset P$ and open $U \subset S^2$ by

$$[K, U] = \{\mathbf{n} \in C^0(P) |\, \mathbf{n}(K) \subset U\}. \tag{2}$$

We note that because P is not compact, the compact-open topology on $C^0(P)$ is distinct from the metric topology on $C^0(P)$, which is induced by the metric

$$d(\mathbf{n}, \mathbf{n}') = \sup_{\mathbf{x} \in P} |\,\mathbf{n}(\mathbf{x}) - \mathbf{n}'(\mathbf{x})|. \tag{3}$$

A path $\mathbf{H}_t \in C^0(P)$ is continuous with respect to the compact-open topology if and only if $\mathbf{H}_t(\mathbf{x})$ is continuous on $P \times [0,1]$. The continuity for $\mathbf{H}_t$ with respect to the metric topology is a stronger condition; in addition to $\mathbf{H}_t(\mathbf{x})$ being continuous on $P \times [0,1]$, $\sup_{x \in P} |\mathbf{H}_t(\mathbf{x}) - \mathbf{H}_{t'}(\mathbf{x})|$ must vanish as t' approaches t.

2 The truncated polyhedron

Let $\mathbf{v}^a$, $a = 1, \ldots, v$, denote the vertices of P. Let E^b, $b = 1, \ldots, e$, denote the edges, and let F^c, $c = 1, \ldots, f$, denote the faces. We regard E^b and F^c as subsets of P.

The truncated polyhedron, denoted $\hat{P}$, is obtained by cleaving P along planes which separate the vertices from each other. Explicitly, let $C^a \subset \mathbb{R}^3$ be a plane which separates the vertex $\mathbf{v}^a$ from the vertices $\mathbf{v}^{b \neq a}$. That is, if $\mathbf{C}^a$ denotes a unit normal to C^a and $\mathbf{c}^a$ is a point in C^a, then $(\mathbf{v}^a - \mathbf{c}^a) \cdot \mathbf{C}^a$ and $(\mathbf{v}^{b \neq a} - \mathbf{c}^a) \cdot \mathbf{C}^a$ have opposite signs. For definiteness, we take $\mathbf{C}^a$ to be outwardly oriented, so that $(\mathbf{v}^a - \mathbf{c}^a) \cdot \mathbf{C}^a > 0$. Let R^a denote the closed half-space given by

$$R^a = \{\mathbf{x} \in \mathbb{R}^3 | (\mathbf{x} - \mathbf{c}^a) \cdot \mathbf{C}^a \leq 0\}. \tag{4}$$

Then the truncated polyhedron $\hat{P}$ is given by

$$\hat{P} = P \cap \left(\cap_{a=1}^{v} R^a\right). \tag{5}$$

$\hat{P}$ is closed and convex.

$\hat{P}$ has two kinds of faces, which we call *cleaved faces* and *truncated faces* (see Fig 1). The cleaved faces, denoted $\hat{C}^a$, are given by the intersections of the planes C^a with P. The truncated faces, denoted $\hat{F}^c$, are given by the intersections of the faces F^c of the original polyhedron P with $\cap_{a=1}^{v} R^a$.

$\hat{P}$ has two kinds of edges, which we call *cleaved edges* and *truncated edges* (see Fig 1). The cleaved edges, denoted by $\hat{B}^{ac}$, are given by the intersections of the cleaved faces $\hat{C}^a$ and the truncated faces $\hat{F}^c$. The truncated edges, denoted by $\hat{E}^b$, are given by the intersections of the original edges E^b with $\cap_{a=1}^{v} R^a$. The boundaries of the cleaved faces consist of cleaved edges. The boundaries of the truncated faces consist of cleaved edges and truncated edges in alternation.

We will say that a continuous unit-vector field on $\hat{P}$ satisfies tangent boundary conditions if, on the truncated face $\hat{F}^c$, the vector field is tangent to

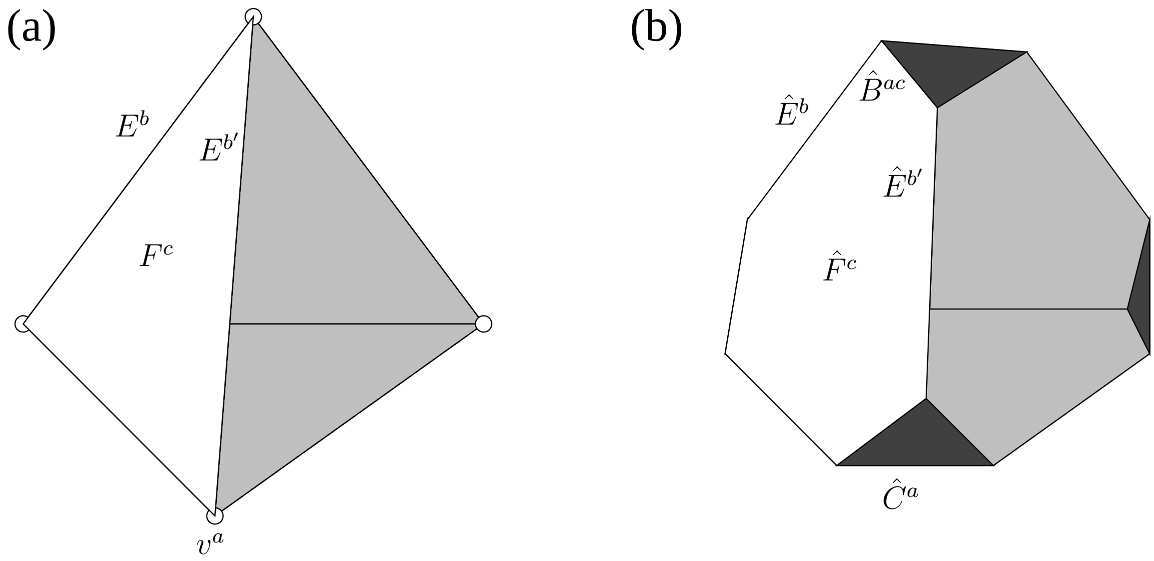

Figure 1: (a) The polyhedron P (b) The cleaved polyhedron $\hat{P}$

$\hat{F}^c$ (note that it need not be tangent on the cleaved faces). Let $C^0(\hat{P})$ denote the space of continuous tangent unit-vector fields on $\hat{P}$. Given $\mathbf{n} \in C^0(P)$, let $\hat{\mathbf{n}}$ denote its restriction to $\hat{P}$. Then $\hat{\mathbf{n}} \in C^0(\hat{P})$.

It turns out that the map $\mathbf{n} \mapsto \hat{\mathbf{n}}$ induces a one-to-one correspondence between homotopy classes of $C^0(P)$ and $C^0(\hat{P})$.

Proposition 2.1. *Given $\mathbf{n}$, $\mathbf{n}' \in C^0(P)$, let $\hat{\mathbf{n}}$, $\hat{\mathbf{n}}' \in C^0(\hat{P})$ denote their restrictions to $\hat{P}$. Then $\mathbf{n} \sim \mathbf{n}'$ if and only if $\hat{\mathbf{n}} \sim \hat{\mathbf{n}}'$.*

Proof. Clearly $\mathbf{n} \sim \mathbf{n}'$ implies $\hat{\mathbf{n}} \sim \hat{\mathbf{n}}'$. For the converse, we introduce maps $\mathbf{N}$, $\mathbf{N}' \in C^0(P)$ which coincide with $\mathbf{n}$, $\mathbf{n}'$ on $\hat{P}$ and are constant along rays in $P - \hat{P}$ through the vertices. These rays are of the form

$$\mathbf{x}^a(r, \mathbf{y}^a) = r\mathbf{y}^a + (1-r)\mathbf{v}^a, \tag{6}$$

where $\mathbf{y}^a \in \hat{C}^a$ and $0 < r < 1$. Every $\mathbf{x} \in P - \hat{P}$ lies on such a ray and uniquely determines the cleaved face $\hat{C}^a$ through which the ray passes as well as $\mathbf{y}^a$ and r. Let $\mathbf{N}$ by given by

$$\mathbf{N}(\mathbf{x}) = \begin{cases} \mathbf{n}(\mathbf{x}), & \mathbf{x} \in \hat{P}, \\ \mathbf{n}(\mathbf{y}^a), & \mathbf{x} = \mathbf{x}^a(r, \mathbf{y}^a). \end{cases} \tag{7}$$

$\mathbf{N}'$ is similarly defined, with $\mathbf{n}$ replaced by $\mathbf{n}'$.

Assuming that $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}'$ are homotopic, it follows that $\mathbf{N}$ and $\mathbf{N}'$ homotopic. Indeed, a homotopy is given by

$$\mathbf{H}_t(\mathbf{x}) = \begin{cases} \hat{\mathbf{H}}_t(\mathbf{x}), & \mathbf{x} \in \hat{P}, \\ \hat{\mathbf{H}}_t(\mathbf{y}^a), & \mathbf{x} = \mathbf{x}^a(r, \mathbf{y}^a), \end{cases} \tag{8}$$

where $\hat{\mathbf{H}}_t$ is a homotopy between $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}'$.

Next we show that $\mathbf{n}$ is homotopic to $\mathbf{N}$. A homotopy $\mathbf{H}_t$ is given by

$$\mathbf{H}_t(\mathbf{x}) = \begin{cases} \mathbf{n}(\mathbf{x}), & \mathbf{x} \in \hat{P}, \\ \mathbf{n}(\mathbf{y}^a), & \mathbf{x} = \mathbf{x}^a(r, \mathbf{y}^a),\ 0 < r < t, \\ \mathbf{n}(\mathbf{x}^a((r-t)/(1-t), \mathbf{y}^a)), & \mathbf{x} = \mathbf{x}^a(r, \mathbf{y}^a),\ t \le r < 1. \end{cases} \tag{9}$$

where $0 \le t \le 1$. Clearly $\mathbf{H}_0 = \mathbf{n}$ and $\mathbf{H}_1 = \mathbf{N}$. It is straightforward to verify that $\mathbf{H}_t(\mathbf{x})$ is continuous for $(\mathbf{x}, t) \in P \times [0, 1]$ and that it satisfies tangent boundary conditions. A similar argument shows that $\mathbf{n}'$ is homotopic to $\mathbf{N}'$. Thus we have a chain of equivalences, $\mathbf{n} \sim \mathbf{N} \sim \mathbf{N}' \sim \mathbf{n}'$, which establishes the required result. □

Thus, the homotopy type of tangent unit-vector fields on P is determined by the homotopy types of their restrictions to the truncated polyhedron $\hat{P}$. Because $\hat{P}$ is closed, the classification of the restricted maps is easier to carry out. For this reason, we determine homotopy classes of $C^0(\hat{P})$ in what follows.

3 Invariants

Given $\hat{\mathbf{n}} \in C^0(\hat{P})$, tangent boundary conditions imply that its values on the truncated edges $\hat{E}^b$ are constant, are tangent to the edges, and therefore are determined up to a sign.

Definition 3.1. *The edge orientation $\mathbf{e}^b(\hat{\mathbf{n}})$ is the value of $\hat{\mathbf{n}}$ on $\hat{E}^b$.*

The edge orientations are obviously homotopy invariants. Under the antipodal map $\hat{\mathbf{n}} \mapsto -\hat{\mathbf{n}}$, the edge orientations obviously change sign.

Kink numbers are relative winding numbers along cleaved edges. Let $\mathbf{z}^{ac}(t)$, $0 \le t \le 1$, denote a continuous parameterisation of the cleaved edge $\hat{B}^{ac}$, positively oriented with respect to the outward normal, denoted $\mathbf{F}^c$, on $\hat{F}^c$. Let $\hat{\mathbf{n}}^{ac}(t) = \hat{\mathbf{n}}(\mathbf{z}^{ac}(t))$. As $\hat{\mathbf{n}}^{ac}(t)$ is tangent to $\hat{F}^c$, its values are related to $\hat{\mathbf{n}}^{ac}(0)$ by a rotation about $\mathbf{F}^c$, which we write as

$$\hat{\mathbf{n}}^{ac}(t) = \mathcal{R}(\mathbf{F}^c, \xi^{ac}(t)) \cdot \hat{\mathbf{n}}^{ac}(0), \tag{10}$$

where $\xi^{ac}(t)$ is the angle of rotation. We take $\xi^{ac}(t)$ to be continuous, and fix it uniquely by taking $\xi^{ac}(0) = 0$.

Let η^{ac}, where $-\pi < \eta^{ac} < \pi$, denote the angle (of smallest magnitude) between $\mathbf{n}^{ac}(0)$ and $\mathbf{n}^{ac}(1)$, so that

$$\hat{\mathbf{n}}^{ac}(1) = \mathcal{R}(\mathbf{F}^c, \eta^{ac}) \cdot \hat{\mathbf{n}}^{ac}(0). \tag{11}$$

(Note that since $\mathbf{n}^{ac}(0)$ and $\mathbf{n}^{ac}(1)$ are parallel to consecutive truncated edges $\hat{E}^b$ and $\hat{E}^{b'}$, they cannot be parallel to each other, so that η^{ac} cannot be a multiple of π). From (10) and (11), $\xi^{ac}(1)$ and η^{ac} differ by an integer multiple of 2π. This integer is the kink number.

Definition 3.2. *The kink number $k^{ac}(\hat{\mathbf{n}})$ is given by*

$$k^{ac}(\hat{\mathbf{n}}) = \frac{1}{2\pi}\left(\xi^{ac}(1) - \eta^{ac}\right). \tag{12}$$

The kink number $k^{ac}(\hat{\mathbf{n}})$ depends continuously on $\hat{\mathbf{n}}$, and therefore is an integer-valued homotopy invariant on $C^0(\hat{P})$. It may be regarded as the degree (winding number) of the map of S^1 to itself obtained by concatenating $\hat{\mathbf{n}}^{ac}(t)$ with a path along which $\hat{\mathbf{n}}^{ac}(1)$ is minimally rotated back to $\hat{\mathbf{n}}^{ac}(0)$ through their common plane.

Equations (10) and (11) remain valid if $\hat{\mathbf{n}}$ is replaced by $-\hat{\mathbf{n}}$. Therefore,

$$k^{ac}(-\hat{\mathbf{n}}) = k^{ac}(\hat{\mathbf{n}}). \tag{13}$$

The kink numbers on each truncated face satisfy the following sum rule:

Proposition 3.1. *Given $\hat{\mathbf{n}} \in C^0(\hat{P})$ and $\hat{F}^c$ a truncated face of $\hat{P}$ with outward normal $\mathbf{F}^c$. Let $q^c(\hat{\mathbf{n}})$ denote the number of pairs of consecutive truncated edges of $\hat{F}^c$ on which $\hat{\mathbf{n}}$ is oppositely oriented with respect to $\mathbf{F}^c$ (ie, $\mathbf{e}^b(\hat{\mathbf{n}}) \cdot \mathbf{e}^{b'}(\hat{\mathbf{n}}) < 0$ for consecutive $\hat{E}^b$ and $\hat{E}^{b'}$). Then*

$$\sum_a{}' k^{ac}(\hat{\mathbf{n}}) = \tfrac{1}{2} q^c(\hat{\mathbf{n}}) - 1, \tag{14}$$

where the sum $\sum_a{}'$ is taken over the cleaved edges $\hat{B}^{ac}$ of $\hat{F}^c$.

Proof. Let $\mathbf{z}^c(t)$, $0 \le t \le 1$, denote a continuous parameterisation of $\partial\hat{F}^c$ (the boundary of $\hat{F}^c$), positively oriented with respect to $\mathbf{F}^c$, with $\mathbf{z}^c(1) = \mathbf{z}^c(0)$. Let $\hat{\mathbf{n}}^c(t) = \hat{\mathbf{n}}(\mathbf{z}^c(t))$, and let

$$\hat{\mathbf{n}}^c(t) = \mathcal{R}(\mathbf{F}^c, \xi^c(t)) \cdot \hat{\mathbf{n}}^c(0), \tag{15}$$

where $\xi^c(t)$ is continuous with $\xi^c(0) = 0$. Along the truncated edges of $\hat{F}^c$, $\xi^c(t)$ is constant. It follows that $\xi^c(1) = \sum_a{}' \xi^{ac}(1)$. But $\xi^c(1)$ is just 2π times the winding number of $\hat{\mathbf{n}}$ around $\partial\hat{F}^c$. Since $\hat{\mathbf{n}}$ is continuous inside $\hat{F}^c$, this winding number vanishes. Therefore

$$\sum_a{}' \xi^{ac}(1) = 0. \tag{16}$$

Taking the sum $\sum_a{}'$ in (12), we conclude that

$$\sum_a{}' k^{ac}(\hat{\mathbf{n}}) = -\sum_a{}' \frac{1}{2\pi}\eta^{ac}. \tag{17}$$

Without loss of generality, we may assume that F^c, the face of the original polyhedron P, is a regular polygon (P can be continuously deformed while remaining convex to make F^c regular). In this case, $\hat{\mathbf{n}}^{ac}(0)$ and $\hat{\mathbf{n}}^{ac}(1)$ are parallel to consecutive edges of a regular polygon. If $\hat{\mathbf{n}}^{ac}(0)$ and $\hat{\mathbf{n}}^{ac}(1)$ are similarly oriented with respect to $\mathbf{F}^c$, then $\eta^{ac} = 2\pi/m$, where m is the number of sides of F^c. If they are oppositely oriented, then $\eta^{ac} = 2\pi/m - \pi$. Substituting into (17), and noting that there are m terms in the sum, we obtain the required result (14). $\square$

Wrapping numbers classify the homotopy type of $\hat{\mathbf{n}}$ on the cleaved faces $\hat{C}^a$. For the explicit definition it will be useful to introduce coordinates on $\hat{C}^a$. Let $\mathbf{z}^a(\phi)$ denote a piecewise-differentiable, 2π-periodic parameterisation of $\partial\hat{C}^a$, positively oriented with respect to the outward normal, denoted $\mathbf{C}^a$, on $\hat{C}^a$. Let $\mathbf{c}^a$ be a point in the interior of $\hat{C}^a$, and let

$$\mathbf{y}^a(\rho, \phi) = \rho\mathbf{z}^a(\phi) + (1 - \rho)\mathbf{c}^a, \tag{18}$$

where $0 \le \rho \le 1$.

To a map $\hat{\mathbf{n}} \in C^0(\hat{P})$, we associate a continuous map $\boldsymbol{\nu}^a$ from D^2, the unit two-disk, to S^2, given by

$$\boldsymbol{\nu}^a(\rho, \phi) = \hat{\mathbf{n}}(\mathbf{y}^a(\rho, \phi)). \tag{19}$$

We construct another continuous map $\boldsymbol{\nu}_0^a : D^2 \to S^2$ as follows. On the boundary of the disk, $\boldsymbol{\nu}_0^a$ is taken to coincide with $\boldsymbol{\nu}^a$. Along radial lines from the boundary to the centre of the disk, $\boldsymbol{\nu}_0^a$ is taken to trace out the shortest geodesic from its value on the boundary to a fixed value, which we denote by $-\mathbf{s}$. (In what follows, we sometimes regard $\mathbf{s}$ as the south pole of S^2, and $-\mathbf{s}$ as the north pole.) Explicitly, let $\mathbf{g}_\rho(-\mathbf{s}, \mathbf{a})$, where $0 \le \rho \le 1$,

denote the shortest geodesic arc from $-\mathbf{s}$ to $\mathbf{a}$, where the parameter ρ is proportional to arclength. Then $\boldsymbol{\nu}_0^a : D^2 \to S^2$ is given by

$$\boldsymbol{\nu}_0^a(\rho, \phi) = \mathbf{g}_\rho(-\mathbf{s}, \hat{\mathbf{n}}(\mathbf{z}^a(\phi))). \tag{20}$$

(20) is well defined provided that the boundary values of $\hat{\mathbf{n}}$ are not antipodal to $-\mathbf{s}$, ie $\hat{\mathbf{n}} \neq \mathbf{s}$ on $\partial\hat{C}^a$. Since, on $\partial\hat{C}^a$, $\hat{\mathbf{n}}$ is tangent to a truncated face, this condition is satisfied provided that

$$\mathbf{s} \cdot \mathbf{F}^c \neq 0, \quad c = 1, \ldots, f. \tag{21}$$

From now on, we assume $\mathbf{s}$ is chosen to satisfy (21). Note that, by construction, $\mathbf{s}$ is not in the image of $\boldsymbol{\nu}_0^a$.

Given two maps $\boldsymbol{\nu}^a, \boldsymbol{\nu}_0^a : D^2 \to S^2$ which coincide on ∂D^2, we may glue them on the boundary to get a continuous map on S^2, which we denote by $\boldsymbol{\nu}^a \circ \boldsymbol{\nu}_0^a$. Explicitly, $\boldsymbol{\nu}^a \circ \boldsymbol{\nu}_0^a : S^2 \to S^2$ is given by

$$(\boldsymbol{\nu}^a \circ \boldsymbol{\nu}_0^a)(x, y, z) = \begin{cases} \boldsymbol{\nu}^a(\rho, \phi), & z \geq 0, \\ \boldsymbol{\nu}_0^a(\rho, \phi), & z < 0, \end{cases} \tag{22}$$

where (ρ, ϕ) are the polar coordinates of (x, y). The wrapping number is the degree of this map.

Definition 3.3. *The wrapping number $w^a(\hat{\mathbf{n}})$ is given by*

$$w^a(\hat{\mathbf{n}}) = \deg(\boldsymbol{\nu}^a \circ \boldsymbol{\nu}_0^a). \tag{23}$$

The wrapping number depends continuously on $\hat{\mathbf{n}}$ (since $\boldsymbol{\nu}^a$ and $\boldsymbol{\nu}_0^a$ do), and therefore is a homotopy invariant.

For $\hat{\mathbf{n}} \in C^1(\hat{P})$ (ie, $\hat{\mathbf{n}}$ is continuously differentiable on $\hat{P}$), we derive an integral formula for the wrapping number. We take $\mathbf{z}^a(\phi)$ to be piecewise-C^1, so that the derivative of $\boldsymbol{\nu}^a \circ \boldsymbol{\nu}_0^a$ is piecewise continuous. Then

$$\deg(\boldsymbol{\nu}^a \circ \boldsymbol{\nu}_0^a) = \frac{1}{4\pi} \int_{S^2} (\boldsymbol{\nu}^a \circ \boldsymbol{\nu}_0^a)^* \, \omega, \tag{24}$$

where ω is the rotationally invariant area-form on S^2, normalised so that $\int_{S^2} \omega = 4\pi$, and $(\boldsymbol{\nu}^a \circ \boldsymbol{\nu}_0^a)^*$ denotes the pull-back. From (22),

$$w^a(\hat{\mathbf{n}}) = \deg(\boldsymbol{\nu}^a \circ \boldsymbol{\nu}_0^a) = \frac{1}{4\pi} \int_{D^2} (\boldsymbol{\nu}^{a*} \omega - \boldsymbol{\nu}_0^{a*} \omega) = \frac{1}{4\pi} \int_{\hat{C}^a} \hat{\mathbf{n}}^* \omega - \frac{1}{4\pi} \int_{D^2} \boldsymbol{\nu}_0^{a*} \omega. \tag{25}$$

By construction, $\boldsymbol{\nu}_0^a$ takes values in $\{S^2 - \mathbf{s}\}$ (the two-sphere with the point $\mathbf{s}$ removed). Let γ denote a one-form on $\{S^2 - \mathbf{s}\}$ for which

$$d\gamma = \omega \text{ on } \{S^2 - \mathbf{s}\}. \tag{26}$$

For example, we may take $\gamma = (1 - \cos\alpha)d\beta$, where (α, β) are spherical polar coordinates on S^2 with south pole at $\mathbf{s}$. Applying Stokes' theorem to the second integral in (25), we get

$$w^a(\hat{\mathbf{n}}) = \frac{1}{4\pi}\left(\int_{\hat{C}^a} \hat{\mathbf{n}}^*\,\omega - \int_{\partial\hat{C}^a} \hat{\mathbf{n}}^*\,\gamma\right). \tag{27}$$

From (27), it is clear that wrapping numbers change sign under the antipodal map $\hat{\mathbf{n}} \to -\hat{\mathbf{n}}$,

$$w^a(-\hat{\mathbf{n}}) = -w^a(\hat{\mathbf{n}}). \tag{28}$$

In fact, (28) holds for all maps in $C^0(\hat{P})$, since any map in $C^0(\hat{P})$ is homotopic to a C^1-map in $C^0(\hat{P})$.

If $\mathbf{s}$ is a regular value of $\hat{\mathbf{n}}$ on $\hat{C}^a$ – that is, if the derivative $d\hat{\mathbf{n}}$ restricted to $\hat{C}^a$ has full rank on $\hat{\mathbf{n}}^{-1}(\mathbf{s})$ – then $\hat{\mathbf{n}}^{-1}(\mathbf{s})$ is a finite set, and we can express the wrapping number as a signed count of preimages $\mathbf{y}_*^a$ of $\mathbf{s}$ on $\hat{C}^a$. We have that

$$\hat{C}^a = \left(\hat{C}^a - \sum_{\mathbf{y}_*^a} U_\epsilon(\mathbf{y}_*^a)\right) + \sum_{\mathbf{y}_*^a} U_\epsilon(\mathbf{y}_*^a), \tag{29}$$

where $U_\epsilon(\mathbf{y}_*^a)$ is an ϵ-neighbourhood of $\mathbf{y}_*^a$. Substituting (29) into the integral formula (27), the contribution from the first term in (29) vanishes due to Stokes' theorem, while each neighbourhood $U_\epsilon(\mathbf{y}_*^a)$ in the second term contributes $\operatorname{sgn}\det d\hat{\mathbf{n}}(\mathbf{y}_*^a)$, where the determinant is computed with respect to positively oriented coordinates on $\hat{C}^a$ and S^2. Then

$$w^a(\hat{\mathbf{n}}) = \sum_{\mathbf{y}_*^a} \operatorname{sgn}\det d\hat{\mathbf{n}}(\mathbf{y}_*^a). \tag{30}$$

Next we use (27) to show that the sum of the wrapping numbers vanishes.

Proposition 3.2. *Given* $\hat{\mathbf{n}} \in C^0(\hat{P})$,

$$\sum_{a=1}^{v} w^a(\hat{\mathbf{n}}) = 0. \tag{31}$$

Proof. We have that

$$\sum_{a=1}^{v} w^a(\hat{\mathbf{n}}) = \sum_{a=1}^{v} \int_{\hat{C}^a} \hat{\mathbf{n}}^* \,\omega - \sum_{a=1}^{v} \int_{\partial \hat{C}^a} \hat{\mathbf{n}}^* \,\gamma. \tag{32}$$

The boundary of the truncated polyhedron $\hat{P}$ is given by

$$\partial \hat{P} = \sum_{a=1}^{v} \hat{C}^a + \sum_{c=1}^{f} \hat{F}^c. \tag{33}$$

Since $\partial(\partial \hat{P}) = 0$,

$$\sum_{a=1}^{v} \partial \hat{C}^a + \sum_{c=1}^{f} \partial \hat{F}^c = 0. \tag{34}$$

The second integral in (32) may then be rewritten as

$$\sum_{a=1}^{v} \int_{\partial \hat{C}^a} \hat{\mathbf{n}}^* \,\gamma = -\sum_{c=1}^{f} \int_{\partial \hat{F}^c} \hat{\mathbf{n}}^* \,\gamma = -\sum_{c=1}^{f} \int_{\hat{F}^c} \hat{\mathbf{n}}^* \,\omega, \tag{35}$$

where in the second equality we have used Stokes' theorem and (26) (this is justified since $\hat{\mathbf{n}} \neq \mathbf{s}$ on $\hat{F}^c$). Substituting (35) into (32), we get

$$\sum_{a=1}^{v} w^a(\hat{\mathbf{n}}) = \left(\sum_{a=1}^{v} \int_{\hat{C}^a} + \sum_{c=1}^{f} \int_{\hat{F}^c} \right) \hat{\mathbf{n}}^* \,\omega = \int_{\partial \hat{P}} \hat{\mathbf{n}}^* \,\omega. \tag{36}$$

Since ω is closed, the last expression vanishes. Therefore

$$\sum_{a=1}^{v} w^a(\hat{\mathbf{n}}) = 0. \tag{37}$$

This result applies to all maps in $C^0(\hat{P})$, as every map in $C^0(\hat{P})$ is homotopic to a C^1-map in $C^0(\hat{P})$. $\square$

The wrapping number depends on the choice of $\mathbf{s} \in S^2$. For $\hat{\mathbf{n}} \in C^1(\hat{P})$, an alternative, convention-independent invariant is the real-valued quantity

$$\Omega^a(\hat{\mathbf{n}}) = \int_{\hat{C}^a} \hat{\mathbf{n}}^* \,\omega = 4\pi w^a(\hat{\mathbf{n}}) + \int_{\partial \hat{C}^a} \hat{\mathbf{n}}^* \,\gamma. \tag{38}$$

We call Ω^a the *trapped area* at the vertex a. It plays a central role in estimates of lower bounds for the energy (1) ([1], [6]).

The second term in the expression (38) for the trapped area can be expressed in terms of the kink numbers and edge orientations, as we now show. We have that

$$\hat{\mathbf{n}}(\partial\hat{C}^a) = \sum_c{}' k^{ac} S^{1c} + K^a, \tag{39}$$

where, in the first term, S^{1c} denotes the unit circle in S^2 normal to $\mathbf{F}^c$, positively oriented with respect to $\mathbf{F}^c$, and the sum $\sum_c{}'$ is taken over the cleaved edges $\hat{B}^{ac}$ of $\partial\hat{C}^a$. From (26), the integral of γ around S^{1c} is given by $-2\pi\,\mathrm{sgn}(\mathbf{F}^c\cdot\mathbf{s})$. The second term in (39), K^a, is the geodesic polygon in S^2 with vertices $\mathbf{e}^{b_1}(\hat{\mathbf{n}}),\ldots\mathbf{e}^{b_m}(\hat{\mathbf{n}})$, where the indices b_r label the truncated edges $\hat{E}^{b_1},\ldots,\hat{E}^{b_m}$ of $\partial\hat{C}^a$, consecutively ordered with respect to the outward normal. Suppose first that K^a has just three vertices, which we denote $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ for convenience. From (26),

$$\int_{K^a}\gamma = A(\mathbf{a},\mathbf{b},\mathbf{c}) - 4\pi\sigma(\mathbf{a},\mathbf{b},\mathbf{c}), \tag{40}$$

where $A(\mathbf{a},\mathbf{b},\mathbf{c})$ is the oriented area of K^a, with the interior of K^a chosen so that $|A(\mathbf{a},\mathbf{b},\mathbf{c})| < 2\pi$, and $\sigma(\mathbf{a},\mathbf{b},\mathbf{c}) = \pm 1, 0$ according to whether $\mathbf{s}$ is outside K^a (in which case $\sigma = 0$) or inside K^a (in which case σ is the orientation of ∂K^a about $\mathbf{s}$). Explicitly, $A(\mathbf{a},\mathbf{b},\mathbf{c})$ is given by

$$A(\mathbf{a},\mathbf{b},\mathbf{c}) = 2\arg((1+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}) + i(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}), \tag{41}$$

where arg is taken between $-\pi$ and π ((41) is equivalent to the standard expression $\alpha+\beta+\gamma-\pi$ for the area of a unit spherical triangle with interior angles α, β and γ.) $\sigma(\mathbf{a},\mathbf{b},\mathbf{c})$ is given by

$$\sigma(\mathbf{a},\mathbf{b},\mathbf{c}) = \begin{cases} \mathrm{sgn}((\mathbf{a}\times\mathbf{b})\cdot\mathbf{s}), & \mathbf{s}\in K^a, \\ 0, & \mathbf{s}\notin K^a. \end{cases} \tag{42}$$

In fact, $\mathbf{s}\in K^a$ if and only if $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{s}$, $(\mathbf{b}\times\mathbf{c})\cdot\mathbf{s}$ and $(\mathbf{c}\times\mathbf{a})\cdot\mathbf{s}$ all have the same sign. If K^a has $m > 3$ vertices, we may represent it as a sum of geodesic triangles K^a_j with vertices $\mathbf{e}^{b_1}(\hat{\mathbf{n}})$, $\mathbf{e}^{b_j}(\hat{\mathbf{n}})$, $\mathbf{e}^{b_{j+1}}(\hat{\mathbf{n}})$, with $2\le j\le m-1$.

These considerations are summarised in the following:

Proposition 3.3. *Given a cleaved face $\hat{C}^a$ with truncated edges $\hat{E}^{b_1},\ldots E^{b_m}$ consecutively ordered with respect to the outward orientation. The trapped area (38) is given by*

$$\Omega^a = 4\pi w^a - 2\pi\sum_c{}' \mathrm{sgn}(\mathbf{F}^c\cdot\mathbf{s})k^{ac} + \sum_{j=2}^{m-1}\left(A(\mathbf{e}^{b_1},\mathbf{e}^{b_j},\mathbf{e}^{b_{j+1}}) - 4\pi\sigma(\mathbf{e}^{b_1},\mathbf{e}^{b_j},\mathbf{e}^{b_{j+1}})\right),$$

where the sum $\sum_c'$ is taken over the cleaved edges $\hat{B}^{ac}$ of $\hat{C}^a$, and A and σ are given by (41) and (42) respectively.

4 Representatives

Let

$$\text{Inv} = \{\mathbf{e}^b, k^{ac}, w^a\} \tag{43}$$

denote the set of homotopy invariants on $C^0(\hat{P})$ defined in Section 3. Let $\mathcal{I} = (\boldsymbol{\epsilon}^b, \kappa^{ac}, \omega^a\}$ denote a set of values of Inv which satisfies the sum rules (14) and (37). In what follows, we construct a representative map $\hat{\mathbf{n}}_{\mathcal{I}} \in C^0(\hat{P})$ for which

$$\text{Inv}(\hat{\mathbf{n}}_{\mathcal{I}}) = \mathcal{I}. \tag{44}$$

We first define $\hat{\mathbf{n}}_{\mathcal{I}}$ on the edges of $\hat{P}$. On the truncated edges, $\hat{\mathbf{n}}_{\mathcal{I}}$ is determined by the edge orientations, $\boldsymbol{\epsilon}^b$.

$$\hat{\mathbf{n}}_{\mathcal{I}}(\mathbf{x}) = \boldsymbol{\epsilon}^b, \;\; \mathbf{x} \in \hat{E}^b. \tag{45}$$

On the cleaved edges, $\hat{\mathbf{n}}_{\mathcal{I}}$ is determined up to homotopy by the edge orientations and the kink numbers, κ^{ac}. Let $\mathbf{z}^{ac}(t)$, $0 \le t \le 1$, denote a parameterisation of $\hat{B}^{ac}$, positively oriented with respect to $\mathbf{F}^c$. Let the endpoints $\mathbf{z}^{ac}(0)$ and $\mathbf{z}^{ac}(1)$ lie on consecutive truncated edges $\hat{E}^b$ and $\hat{E}^{b'}$ respectively. Let $\eta^{ac} \in (-\pi, \pi)$ denote the angle from $\boldsymbol{\epsilon}^b$ to $\boldsymbol{\epsilon}^{b'}$, as in (11). Then on $\hat{B}^{ac}$, we take $\hat{\mathbf{n}}_{\mathcal{I}}$ to be given by

$$\hat{\mathbf{n}}_{\mathcal{I}}(\mathbf{z}^{ac}(t)) = \mathcal{R}(\mathbf{F}^c, (\eta^{ac} + 2\pi\kappa^{ac})t) \cdot \boldsymbol{\epsilon}^b. \tag{46}$$

To extend $\hat{\mathbf{n}}_{\mathcal{I}}$ to the faces of $\hat{P}$, it is convenient to introduce polygonal-polar coordinates. Let $\mathbf{f}^c$ be a point in the interior of the truncated face $\hat{F}^c$. We parameterise $\hat{F}^c$ by

$$\mathbf{y}^c(\rho, \mathbf{z}^c) = \rho\mathbf{z}^c + (1-\rho)\mathbf{f}^c, \tag{47}$$

where $0 \le \rho \le 1$ and $\mathbf{z}^c \in \partial\hat{F}^c$. By a radial chord, we mean the segment obtained by taking $\mathbf{z}^c$ fixed in (47), and letting ρ vary between 0 and 1. Similarly, let $\mathbf{c}^a$ be a point in the interior of the cleaved face $\hat{C}^a$. We parameterise $\hat{C}^a$ by

$$\mathbf{y}^a(\rho, \mathbf{z}^a) = \rho\mathbf{z}^c + (1-\rho)\mathbf{c}^a, \tag{48}$$

Radial chords on $\hat{C}^a$ are defined as for $\hat{F}^c$.

On $\hat{F}^c$, we define $\hat{\mathbf{n}}_{\mathcal{I}}$ along radial chords by contracting its boundary values to a constant. Explicitly, we note that (45) and (46) determine $\hat{\mathbf{n}}_{\mathcal{I}}$ on $\partial\hat{F}^c$. We regard $\hat{\mathbf{n}}_{\mathcal{I}}$ on $\partial\hat{F}^c$ as a continuous map of S^1 to itself (the image lies in S^{1c}, the great circle orthogonal to $\hat{F}^c$). Since the kink numbers κ^{ac} satisfy the sum rule (14), this map has zero winding number, and therefore

is contractible. That is, there exists a continuous unit-vector field $\hat{\mathbf{h}}_t(\mathbf{z}^c)$ tangent to $\hat{F}^c$ such that $\hat{\mathbf{h}}_0^c(\mathbf{z}^c) = \hat{\mathbf{n}}_\mathcal{I}(\mathbf{z}^c)$ and $\hat{\mathbf{h}}_1^c$ is constant. Let

$$\hat{\mathbf{n}}_\mathcal{I}(\mathbf{y}^c(\rho, \mathbf{z}^c)) = \hat{\mathbf{h}}_\rho^c(\mathbf{z}^c). \tag{49}$$

On $\hat{C}^a$, we note that (46) determines the values of $\hat{\mathbf{n}}_\mathcal{I}$ on $\partial\hat{C}^a$, where $\rho = 1$. We define $\hat{\mathbf{n}}_\mathcal{I}$ for $\frac{1}{2} \le \rho < 1$ by contracting its boundary values along shortest geodesics on S^2 to $-\mathbf{s}$.

$$\hat{\mathbf{n}}_\mathcal{I}(\mathbf{y}^a(\rho, \mathbf{z}^a)) = g_{2\rho-1}(-\mathbf{s}, \hat{\mathbf{n}}_\mathcal{I}(\mathbf{y}^a(1, \mathbf{z}^a))), \quad \tfrac{1}{2} \le \rho < 1, \tag{50}$$

where $g_\tau(-\mathbf{s}, \mathbf{a})$, $0 \le \tau \le 1$, denotes the shortest geodesic from $-\mathbf{s}$ to $\mathbf{a}$ (as in (20)). For $\rho \le \frac{1}{2}$, we insert a covering of S^2 with multiplicity given by the wrapping number ω^a. Explicitly, let $\mathbf{z}^a(\phi)$ be a 2π-periodic parameterisation of $\partial\hat{C}^a$, and let

$$\hat{\mathbf{n}}_\mathcal{I}(\mathbf{y}^a(\rho, \mathbf{z}^a(\phi))) = \sin 2\pi\rho \cos \omega^a\phi\, \boldsymbol{\xi} + \sin 2\pi\rho \sin \omega^a\phi\, \boldsymbol{\eta} + \cos 2\pi\rho\, \mathbf{s},\ 0 \le \rho < \tfrac{1}{2}, \tag{51}$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are orthonormal vectors in the plane perpendicular to $\mathbf{s}$ with $\boldsymbol{\xi} \times \boldsymbol{\eta} = -\mathbf{s}$. Let (α, β) denote polar coordinates on S^2 with south pole at $\mathbf{s}$. Identifying S^2 with the region $\rho \le \frac{1}{2}$ on $\hat{C}^a$ via $\rho = (\pi - \alpha)/2\pi$, $\mathbf{z}^a = \mathbf{z}^a(\beta)$, then (51) corresponds to the S^2-map $(\alpha, \beta) \mapsto (\alpha, \omega^a\beta)$, which has degree ω^a. It is readily verified from (23) that $w^a(\hat{\mathbf{n}}_\mathcal{I}) = \omega^a$.

We extend $\hat{\mathbf{n}}_\mathcal{I}$ to the interior of $\hat{P}$ along radial lines by contracting its boundary values to a constant. Explicitly, we note that (49), (50) and (51) determine $\hat{\mathbf{n}}_\mathcal{I}$ on $\partial\hat{P}$. From (36), the integral of $\hat{\mathbf{n}}_\mathcal{I}^*\omega$ over $\partial\hat{P}$ is given by the sum of the wrapping numbers ω^a. By assumption, this sum vanishes, so that

$$\int_{\partial\hat{P}} \hat{\mathbf{n}}_\mathcal{I}^*\omega = 0 \tag{52}$$

(we can take $\hat{\mathbf{n}}_\mathcal{I}$ to be piecewise-differentiable on $\partial\hat{P}$, so that $\hat{\mathbf{n}}_\mathcal{I}^*\omega$ is piecewise-continuous). Regarding $\partial\hat{P}$ as a topological two-sphere, we may regard $\hat{\mathbf{n}}_\mathcal{I}$ on $\partial\hat{P}$ as a degree-zero map on S^2. There exists a contraction to a constant map. Let $\hat{\mathbf{h}}_t : \partial\hat{P} \to S^2$, where $0 \le t \le 1$, be such a contraction, ie $\hat{\mathbf{h}}_t$ is continuous, $\hat{\mathbf{h}}_0 = \hat{\mathbf{n}}_\mathcal{I}$ and $\hat{\mathbf{h}}_1 = \mathbf{s}$, constant. Let $\mathbf{p}$ be a point in the interior of $\hat{P}$, and let

$$\mathbf{x}(r, \mathbf{y}) = r\mathbf{y} + (1 - r)\mathbf{p}, \tag{53}$$

where $0 \le r \le 1$. Then we define $\hat{\mathbf{n}}_\mathcal{I}$ in $\hat{P}$ by

$$\hat{\mathbf{n}}_\mathcal{I}(\mathbf{x}(r, \mathbf{y}))) = \hat{\mathbf{h}}_r(\mathbf{y}). \tag{54}$$

Let $\mathbf{c}^{a_*}$ denote the interior point of the cleaved face $\hat{C}^{a_*}$. Setting $\rho = 0$ in (51), we see that $\hat{\mathbf{n}}_\mathcal{I}(\mathbf{c}^{a_*}) = \mathbf{s}$. Without loss of generality, and for future

convenience, we choose the homotopy $\hat{\mathbf{h}}_t$ so that $\hat{\mathbf{h}}_t(\mathbf{c}^{a_*}) = \mathbf{s}$ for all $0 \le t \le 1$. Therefore, from (54),

$$\hat{\mathbf{n}}_{\mathcal{I}}(\mathbf{x}(\rho, \mathbf{c}^{a_*})) = \mathbf{s}. \tag{55}$$

We note that the construction of $\hat{\mathbf{n}}_{\mathcal{I}}$ is not completely explicit, in that we make use of the contractibility of degree-zero maps on S^1 and S^2 without specifying these contractions explicitly. An explicit prescription for these contractions (which is valid for all S^n) is described by, eg, Whitehead [9] (of course, for S^1, the contraction is easily constructed).

5 Classification

Our main result is that the invariants, Inv, classify maps in $C^0(\hat{P})$ up to homotopy.

Theorem 1. *Let $\hat{\mathbf{n}}, \hat{\mathbf{n}}' \in C^0(\hat{P})$. Then $\hat{\mathbf{n}} \sim \hat{\mathbf{n}}'$ if and only if* $\mathrm{Inv}(\hat{\mathbf{n}}) = \mathrm{Inv}(\hat{\mathbf{n}}')$.

Proof. Since $\mathrm{Inv}(\hat{\mathbf{n}})$ is homotopy invariant, it is clear that $\hat{\mathbf{n}} \sim \hat{\mathbf{n}}'$ only if $\mathrm{Inv}(\hat{\mathbf{n}}) = \mathrm{Inv}(\hat{\mathbf{n}}')$. For the converse, it suffices to show that $\hat{\mathbf{n}}$ is homotopic to the representative map $\hat{\mathbf{n}}_{\mathcal{I}}$, where $\mathcal{I} = \mathrm{Inv}(\hat{\mathbf{n}})$.

It will be convenient to use the polyhedral-polar coordinates $\mathbf{x}(r, \mathbf{y})$ on $\hat{P}$ given by (53), where $0 \le r \le 1$ and $\mathbf{y} \in \partial\hat{P}$. The sets r = constant interpolate between the boundary $\partial\hat{P}$ ($r = 1$) and the interior point $\mathbf{p}$ ($r = 0$). Let $\hat{P}(a, b)$ denote the polyhedral shell $a \le r \le b$. With an abuse of notation, we shall sometimes write, for the sake of brevity, $\hat{\mathbf{n}}(r, \mathbf{y})$, rather than $\hat{\mathbf{n}}(\mathbf{x}(r, \mathbf{y}))$, and similarly for other maps in $C^0(\hat{P})$.

To show that $\hat{\mathbf{n}} \sim \hat{\mathbf{n}}_{\mathcal{I}}$, we argue as follows. First, we deform $\hat{\mathbf{n}}$ to a map $\hat{\mathbf{n}}_1$ which coincides with a radially scaled copy of $\hat{\mathbf{n}}_{\mathcal{I}}$ on the outer shell $\hat{P}(\frac{1}{2}, 1)$ and which is constant, equal to $\mathbf{s}$, on the inner shell $\hat{P}(\epsilon, \frac{1}{2})$, where $\epsilon > 0$ is specified below. The dependence of $\hat{\mathbf{n}}_1$ on the original map $\hat{\mathbf{n}}$ is confined to the polyhedral bubble $\hat{P}(0, \epsilon)$. Then, we create a radial channel through the outer shell, inside of which the map is made to be constant, equal to $\mathbf{s}$. The polyhedral bubble is made to evaporate through this channel. The channel is then removed, leaving a map which is a radially scaled copy of $\hat{\mathbf{n}}_{\mathcal{I}}$ on $\hat{P}(\frac{1}{2}, 1)$ and which is constant, equal to $\mathbf{s}$, on $\hat{P}(0, \frac{1}{2})$. A final rescaling produces $\hat{\mathbf{n}}_{\mathcal{I}}$. A schematic description of these deformations is shown in Fig 2. Details of the argument follow below.

Without loss of generality, we may assume that $\hat{\mathbf{n}}$ coincides with $\hat{\mathbf{n}}_{\mathcal{I}}$ on $\partial\hat{P}$; this is demonstrated in the following section (see Prop 6.1). Then for

any $0 < \epsilon < \frac{1}{2}$, $\hat{\mathbf{n}}$ is homotopic to a map $\hat{\mathbf{n}}_1 \in C^0(\hat{P})$ given by

$$\hat{\mathbf{n}}_1(r, \mathbf{y}) = \begin{cases} \hat{\mathbf{n}}_{\mathcal{I}}(2r-1, \mathbf{y}), & \frac{1}{2} \le r \le 1, \\ \mathbf{s}, & \epsilon \le r < \frac{1}{2} \end{cases} \tag{56}$$

for $\epsilon \le r \le 1$. Note that, from (55), $\hat{\mathbf{n}}_{\mathcal{I}}(0, \mathbf{y}) = \mathbf{s}$, so that $\hat{\mathbf{n}}_1$ is continuous at $r = \frac{1}{2}$. For $r < \epsilon$, $\hat{\mathbf{n}}_1$ is given by

$$\hat{\mathbf{n}}_1(r, \mathbf{y}) = \begin{cases} \hat{\mathbf{n}}_{\mathcal{I}}(2(\epsilon - r)/\epsilon, \mathbf{y}), & \frac{1}{2}\epsilon \le r < \epsilon, \\ \hat{\mathbf{n}}(2r/\epsilon, \mathbf{y}), & 0 \le r < \frac{1}{2}\epsilon. \end{cases} \tag{57}$$

See Fig 2(b). In fact, the particular form for $r \le \epsilon$ will not concern us in what follows. A homotopy between $\hat{\mathbf{n}}_{\mathcal{I}}$ and $\hat{\mathbf{n}}_1$ is given by

$$\mathbf{H}_t(r, \mathbf{y}) = \begin{cases} \hat{\mathbf{n}}_{\mathcal{I}}(\sigma(r), \mathbf{y}), & 1 - \frac{1}{2}t \le r \le 1, \\ \hat{\mathbf{n}}_{\mathcal{I}}(1-t, \mathbf{y}), & 1 - (1-\epsilon)t \le r < 1 - \frac{1}{2}t, \\ \hat{\mathbf{n}}_{\mathcal{I}}(\tau_t(r), \mathbf{y}), & 1 - (1 - \frac{1}{2}\epsilon)t \le r < 1 - (1-\epsilon)t, \\ \hat{\mathbf{n}}(\upsilon_t(r), \mathbf{y}), & r < 1 - (1-\epsilon/2)t, \end{cases} \tag{58}$$

where

$$\begin{aligned} \sigma(r) &= 2r - 1, \\ \tau_t(r) &= 1 + 2((1-r) - (1 - \tfrac{1}{2}\epsilon)t)/\epsilon, \\ \upsilon_t(r) &= r/(1 - (1 - \tfrac{1}{2}\epsilon)t). \end{aligned} \tag{59}$$

Consider the set T given by

$$T = \{\mathbf{x}(r, \mathbf{y}^{a_*}(\rho, \mathbf{z}^{a_*})) | \, r \ge \tfrac{1}{2}, \rho \le \tfrac{1}{2}\}, \tag{60}$$

where $\mathbf{y}^{a_*}(\rho, \mathbf{z}^{a_*})$ denotes the polygonal-polar coordinates (48) on $\hat{C}^{a_*}$. T represents a channel in the outer shell $\hat{P}(\frac{1}{2}, 1)$ through the cleaved face $\hat{C}^{a_*}$. The central axis of T, where $\rho = 0$, is given by $(1-r)\mathbf{c}^{a_*} + r\mathbf{p}$, $r \ge \frac{1}{2}$. From (55) and (56), it follows that $\hat{\mathbf{n}}_1 = \mathbf{s}$ along this axis. We show that $\hat{\mathbf{n}}_1$ is homotopic to a map $\hat{\mathbf{n}}_2$ which is equal to $\mathbf{s}$ throughout T, and which coincides with $\hat{\mathbf{n}}_1$ for $r < \frac{1}{2}$ and for $\mathbf{y} \notin \hat{C}^{a_*}$. A homotopy $\hat{\mathbf{H}}_t(r, \mathbf{y})$ is given by $\hat{\mathbf{n}}_1(r, \mathbf{y})$ for $r < \frac{1}{2}$ or $\mathbf{y} \notin \hat{C}^{a_*}$, and for $r \ge \frac{1}{2}$ and $\mathbf{y} \in \hat{C}^{a_*}$ by

$$\hat{\mathbf{H}}_t(r, \mathbf{y}^{a_*}(\rho, \mathbf{z}^{a_*})) = \begin{cases} \hat{\mathbf{n}}_1(r, \mathbf{y}^{a_*}((2\rho - t)/(2-t), \mathbf{z}^{a_*})), & t/2 < \rho \le 1, \\ \mathbf{s}, & 0 \le \rho \le t/2, \end{cases} \tag{61}$$

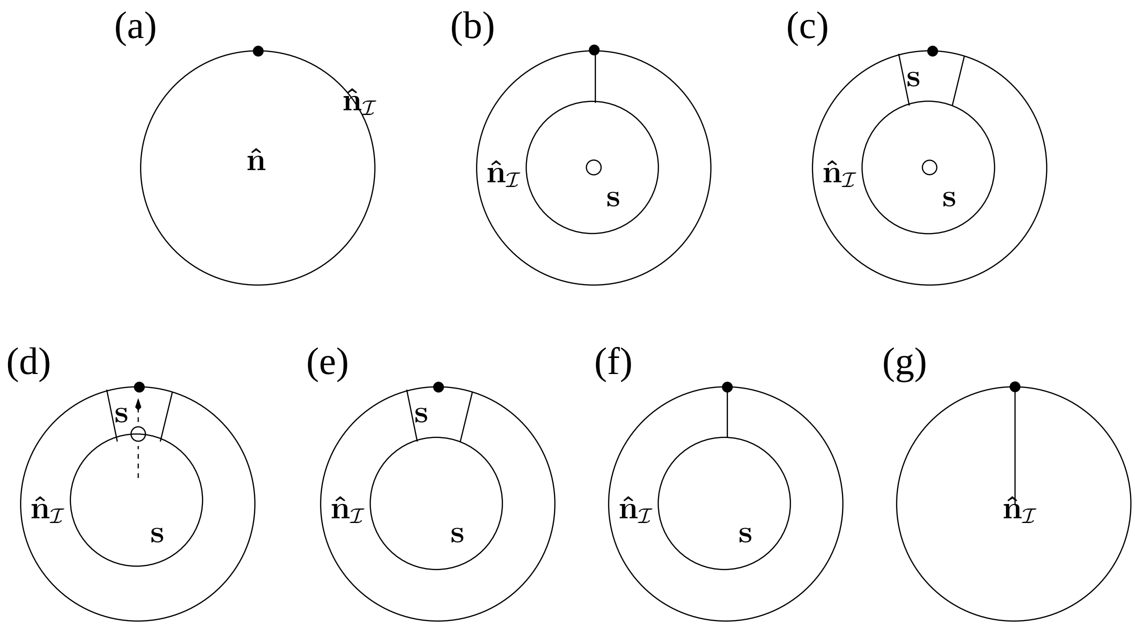

Figure 2: Homotopy from $\hat{\mathbf{n}}$ to $\hat{\mathbf{n}}_{\mathcal{I}}$. Polyhedral shells $\hat{P}(a,b)$ are represented schematically as spherical shells. (a) $\hat{\mathbf{n}}$ coincides with $\hat{\mathbf{n}}_{\mathcal{I}}$ on $\partial\hat{P}$. The marked point is $\mathbf{c}^{a_*}$, where $\hat{\mathbf{n}}_{\mathcal{I}} = \mathbf{s}$. (b) $\hat{\mathbf{n}}_1$. Note that $\hat{\mathbf{n}}_1 = \mathbf{s}$ along the outer half of the ray from $\mathbf{c}^{a_*}$ to the centre. (c) $\hat{\mathbf{n}}_2$ is equal to $\mathbf{s}$ in the channel. (d) The polyhedral bubble, $P(0,\epsilon)$, is floated through the channel. (e) $\hat{\mathbf{n}}_3$ (f) The channel is removed to obtain $\hat{\mathbf{n}}_4$. (g) $\hat{\mathbf{n}}_{\mathcal{I}}$

where $\mathbf{z}^{a_*} \in \partial\hat{C}^{a_*}$. Let $\hat{\mathbf{n}}_2 = \hat{\mathbf{H}}_1$. Then, for $r \geq \frac{1}{2}$,

$$\hat{\mathbf{n}}_2(r, \mathbf{y}^{a_*}(\rho, \mathbf{z}^{a_*})) = \begin{cases} \hat{\mathbf{n}}_1(r, \mathbf{y}^{a_*}(2\rho - 1, \mathbf{z}^{a_*})), & \frac{1}{2} < \rho \leq 1, \\ \mathbf{s}, & 0 \leq \rho \leq \frac{1}{2}. \end{cases} \tag{62}$$

$\hat{\mathbf{n}}_2$ is constant, equal to $\mathbf{s}$, in the inner shell $\hat{P}(\epsilon, \frac{1}{2})$ as well as in T. See Fig. 2(c).

Next we deform $\hat{\mathbf{n}}_2$ so that it is constant, equal to $\mathbf{s}$, throughout the whole inner polyhedron $\hat{P}(0, \frac{1}{2})$. This is accomplished by displacing the polyhedral bubble in which $\hat{\mathbf{n}}_1$ is varying from $\hat{P}(0, \epsilon)$ through the shell $\hat{P}(\epsilon, \frac{1}{2})$ and then through the channel T. Let $\mathbf{u}$ be parallel to the axis of T, ie proportional to $\mathbf{c}^{a_*} - \mathbf{p}$, with $|\mathbf{u}|$ sufficiently large so that

$$\left\{\hat{P}(0, \epsilon) + \mathbf{u}\right\} \cap \hat{P} = \emptyset. \tag{63}$$

Choose ϵ sufficiently small so that

$$\left\{\hat{P}(0, \epsilon) + t\mathbf{u}\right\} \cap \hat{P} \subset \hat{P}(0, \tfrac{1}{2}) \cup T, \quad 0 \leq t \leq 1. \tag{64}$$

Let

$$\hat{\mathbf{H}}_t(\mathbf{x}) = \begin{cases} \hat{\mathbf{n}}_2(\mathbf{x} - t\mathbf{u}), & \mathbf{x} \in \left\{\hat{P}(0, \epsilon) + t\mathbf{u}\right\} \cap \hat{P}, \\ \mathbf{s}, & \mathbf{x} \in \hat{P}(0, \epsilon) \text{ and } \mathbf{x} \notin \left\{\hat{P}(0, \epsilon) + t\mathbf{u}\right\}, \\ \hat{\mathbf{n}}_2(\mathbf{x}), & \text{otherwise.} \end{cases} \tag{65}$$

See Fig. 2(d). (64) guarantees that $\hat{\mathbf{H}}_t(\mathbf{x})$ is continuous, as $\hat{\mathbf{n}}_2$ is continuous and is constant, equal to $\mathbf{s}$, throughout $P(\epsilon, \frac{1}{2}) \cup T$. Let $\hat{\mathbf{n}}_3 = \hat{\mathbf{H}}_1$. From (63) and (62), it follows that $\hat{\mathbf{n}}_3$ is constant, equal to $\mathbf{s}$, on $\hat{P}(0, \epsilon)$ and that it coincides with $\hat{\mathbf{n}}_2$ in $\hat{P}(\frac{1}{2}, 1)$. See Fig. 2(e). By applying the inverse of the homotopy (61), with $\hat{\mathbf{n}}_1$ replaced by $\hat{\mathbf{n}}_3$, we can collapse the channel T to obtain a map $\hat{\mathbf{n}}_4$ (see Fig. 2(f)) given by

$$\hat{\mathbf{n}}_4(r, \mathbf{y}) = \begin{cases} \hat{\mathbf{n}}_{\mathcal{I}}(2r - 1, \mathbf{y}), & \frac{1}{2} \leq r \leq 1, \\ \mathbf{s}, & r < \frac{1}{2}. \end{cases} \tag{66}$$

Then

$$\hat{\mathbf{H}}_t(r, \mathbf{y}) = \begin{cases} \hat{\mathbf{n}}_{\mathcal{I}}((2r - (1-t))/(1+t), \mathbf{y}), & \frac{1}{2}(1-t) \leq r \leq 1, \\ \mathbf{s}, & \rho < \frac{1}{2}(1-t) \end{cases} \tag{67}$$

describes a homotopy of $\hat{\mathbf{n}}_4$ to $\hat{\mathbf{n}}_{\mathcal{I}}$. $\square$

6 Surface homotopies

An intermediate step in the proof of Theorem 1 is the fact that maps in $C^0(\hat{P})$ can be deformed to coincide with their associated representative maps on $\partial\hat{P}$. This is summarised by the following:

Proposition 6.1. *Let $\hat{\mathbf{n}} \in C^0(\hat{P})$, with $\mathcal{I} = \mathrm{Inv}(\hat{\mathbf{n}})$. Then $\hat{\mathbf{n}}$ is homotopic to a map $\hat{\mathbf{n}}'$ for which $\hat{\mathbf{n}}' = \hat{\mathbf{n}}_{\mathcal{I}}$ on $\partial\hat{P}$.*

To prove Proposition 6.1, we make use of the fact that deformations of $\hat{\mathbf{n}}$ on the edges of $\hat{P}$ can be extended to deformations of $\hat{\mathbf{n}}$ on the faces, and, similarly, deformations of $\hat{\mathbf{n}}$ on the faces of $\hat{P}$ can be extended to deformations of $\hat{\mathbf{n}}$ on $\hat{P}$ itself. For completeness, we give an argument below which covers both cases (of course, a similar result holds generally on manifolds with boundary).

Lemma 6.1. *Let $Q \subset \mathbb{R}^k$ be compact and convex with boundary ∂Q, and let S be a topological space with subspace T. Let $C^0(Q)$ denote the space of continuous maps from Q to S which map ∂Q to T, and let $C^0(\partial Q)$ denote the space of continuous maps of ∂Q to T. Given $n \in C^0(Q)$, let $\partial n \in C^0(\partial Q)$ denote its restriction to ∂Q. Suppose that ∂n is homotopic to $\nu' \in C^0(\partial Q)$. Then n is homotopic to some $n' \in C^0(Q)$ with $\partial n' = \nu'$.*

Proof. Introduce polygonal-polar coordinates on Q. Ie, let $\mathbf{q}$ be a point in the interior of Q, and let $\mathbf{u}(\lambda, \mathbf{v}) = \lambda\mathbf{v} + (1-\lambda)\mathbf{q}$, where $0 \le \lambda \le 1$ and $\mathbf{v} \in \partial Q$. Given $n \in C^0(Q)$, we write, by an abuse of notation but for the sake of brevity, $n(\lambda, \mathbf{v})$ rather than $n(\mathbf{u}(\lambda, \mathbf{v}))$, and similarly for other maps in $C^0(Q)$. Let h_t be a homotopy from ∂n to ν'. Let H_t be given by

$$H_t(\rho, \mathbf{v}) = \begin{cases} h_{2\rho+t-2}(\mathbf{v}), & 1 - \frac{1}{2}t < \rho \le 1, \\ n(\rho/(1-\frac{1}{2}t), \mathbf{v}), & \rho \le 1 - \frac{1}{2}t. \end{cases} \tag{68}$$

Let $n' = H_1$. Then n is homotopic to n', and $\partial n' = \nu'$. $\square$

Proof of Proposition 6.1. Let $C^0(\partial\hat{P})$ denote the space of continuous tangent unit-vector fields on the boundary of $\hat{P}$ (so that $\hat{\mathbf{n}}(\mathbf{y})$ is tangent to $\partial\hat{P}$ at $\mathbf{y}$). Given $\hat{\mathbf{n}} \in C^0(\hat{P})$, let $\partial\hat{\mathbf{n}} \in C^0(\partial\hat{P})$ denote its restriction to $\partial\hat{P}$.

From Lemma 6.1, it suffices to show that

$$\partial\hat{\mathbf{n}} \sim \partial\hat{\mathbf{n}}_{\mathcal{I}}, \tag{69}$$

where we have the usual notion of homotopic equivalence in $C^0(\partial\hat{C})$. We establish (69) in two steps, first deforming $\partial\hat{\mathbf{n}}$ to coincide with $\partial\hat{\mathbf{n}}_{\mathcal{I}}$ on the

edges of $\partial\hat{P}$, and then deforming it further to coincide with $\partial\hat{\mathbf{n}}_{\mathcal{I}}$ on the faces of $\partial\hat{P}$.

Since $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}_{\mathcal{I}}$ have the same edge orientations (ie, $\mathbf{e}^b(\hat{\mathbf{n}}) = \epsilon^b$), they coincide on truncated edges, and therefore coincide on the endpoints of the cleaved edges $\hat{B}^{ac}$. Since $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}_{\mathcal{I}}$ have the same kink numbers, there is a homotopy between the restrictions of $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}_{\mathcal{I}}$ to the cleaved edges. (Explicitly, if, on $\hat{B}^{ac}$, $\hat{\mathbf{n}}$ is represented by an angle $\theta^{ac}(s)$ in the plane tangent to $\hat{F}^c$, with $0 \le s \le 1$, and $\hat{\mathbf{n}}_{\mathcal{I}}$ is similarly represented by $\theta'^{ac}(s)$ with $\theta'^{ac}(0) = \theta^{ac}(0)$, then $k^{ac} = \kappa^{ac}$ implies that $\theta'^{ac}(1) = \theta^{ac}(1)$, and a homotopy is given by $(1-t)\theta^{ac}(s) + t\theta'^{ac}(s)$). By Lemma 6.1, these homotopies on $\hat{B}^{ac}$ can be extended to homotopiwhich will be convenient in what follows.

es on the faces of $\hat{P}$, and therefore to a homotopy $\hat{\mathbf{h}}_t$ on $\partial\hat{P}$. Let $\hat{\boldsymbol{\nu}}' = \hat{\mathbf{h}}_1$. By construction, $\hat{\boldsymbol{\nu}}'$ coincides with $\partial\hat{\mathbf{n}}_{\mathcal{I}}$ on the edges of $\partial\hat{P}$.

Next, we construct homotopies from $\hat{\boldsymbol{\nu}}'$ to $\partial\hat{\mathbf{n}}_{\mathcal{I}}$ on the faces of $\hat{P}$. On the truncated face $\hat{F}^c$, $\hat{\boldsymbol{\nu}}'$ may be represented by an angle $\theta'^c(\mathbf{y}^c)$ in the plane tangent to $\hat{F}^c$. The sum rule (14) ensures that $\theta'^c(\mathbf{y}^c)$ is continuous. $\partial\hat{\mathbf{n}}_{\mathcal{I}}$ may be similarly represented by $\theta^c(\mathbf{y}^c)$. By construction, $\theta'^c(\mathbf{y}^c)$ and $\theta^c(\mathbf{y}^c)$ agree on $\partial\hat{F}^c$ up to addition of a multiple of 2π, which we can take to vanish. A homotopy between them on $\hat{F}^c$ is given by $(1-t)\theta'^c(\mathbf{y}^c) + t\theta^c(\mathbf{y}^c)$.

Homotopies from $\hat{\boldsymbol{\nu}}'$ to $\partial\hat{\mathbf{n}}_{\mathcal{I}}$ on the cleaved faces may be constructed as follows. Let $\mathbf{y}^a(\rho, \mathbf{z}^a)$ be the polygonal-polar coordinates on $\hat{C}^a$ given by (48), with $0 \le \rho \le 1$ and $\mathbf{z}^a \in \partial\hat{C}^a$. We first deform $\hat{\boldsymbol{\nu}}'$ so that it agrees with $\partial\hat{\mathbf{n}}_{\mathcal{I}}$ for $\rho \ge \frac{1}{2}$. A homotopy is given by

$$\hat{\mathbf{h}}^a_t(\rho, \mathbf{z}^a) = \begin{cases} \partial\hat{\mathbf{n}}_{\mathcal{I}}(2\rho - 1, \mathbf{z}^a), & 1 - \frac{1}{2}t < \rho \le 1, \\ \partial\hat{\mathbf{n}}_{\mathcal{I}}(5 - 4\rho - 3t, \mathbf{z}^a), & 1 - \frac{3}{4}t < \rho \le 1 - \frac{1}{2}t, \\ \hat{\boldsymbol{\nu}}'(\rho/(1 - \frac{3}{4}t), \mathbf{z}^a), & 0 \le \rho \le 1 - \frac{3}{4}t. \end{cases} \tag{70}$$

Let $\hat{\boldsymbol{\nu}}'' = \hat{\mathbf{h}}^a_1$. Then $\hat{\boldsymbol{\nu}}''$ coincides with $\hat{\mathbf{n}}_{\mathcal{I}}$ for $\rho \ge \frac{1}{2}$.

The region $\rho \le \frac{1}{2}$ on $\hat{C}^a$ is a topological two-disk. On the boundary, where $\rho = \frac{1}{2}$, $\hat{\boldsymbol{\nu}}''$ and $\partial\hat{\mathbf{n}}_{\mathcal{I}}$ are both constant, equal to $-\mathbf{s}$ (cf (50) and (20).) By identifying points on the boundary, we may regard $\hat{\boldsymbol{\nu}}''$ and $\hat{\mathbf{n}}_{\mathcal{I}}$ as maps on S^2 which preserve a marked point $-\mathbf{s}$. The fact that $w^a(\hat{\mathbf{n}}_{\mathcal{I}}) = \omega^a$ implies that these maps have the same degree, and therefore are homotopic. Thus there exists a homotopy on $\rho \le \frac{1}{2}$ which takes $\hat{\boldsymbol{\nu}}''$ to $\partial\hat{\mathbf{n}}_{\mathcal{I}}$ and which is equal to $-\mathbf{s}$ for $\rho = \frac{1}{2}$. This establishes a homotopy between $\hat{\boldsymbol{\nu}}''$ and $\partial\hat{\mathbf{n}}_{\mathcal{I}}$ on $\hat{C}^a$.

Together, the homotopies on truncated faces and cleaved faces give a homotopy from $\hat{\boldsymbol{\nu}}''$ to $\partial\hat{\mathbf{n}}_{\mathcal{I}}$. The chain of equivalences $\partial\hat{\mathbf{n}} \sim \hat{\boldsymbol{\nu}}' \sim \hat{\boldsymbol{\nu}}'' \sim \partial\hat{\mathbf{n}}_{\mathcal{I}}$ in $C^0(\partial\hat{P})$ gives the required result.

□

7 Concluding remarks

The problem considered here may be generalised to $n > 3$ dimensions. Generalisations suggested by liquid crystal applications include normal boundary conditions (ie, on the faces of P, $\mathbf{n}$ is required to be orthogonal to the faces), and periodic boundary conditions on a cubic domain from which a polyhedral domain has been excised (this corresponds to an array of liquid crystal cells with polyhedral geometries). It may be our results apply to nonconvex polyhedra as well.

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